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CONDITIONS FOR EXISTENCE OF EXACT SOLUTIONS OF THE PROBLEM OF A FUSING WEDGE

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A well-defined condition, determining the values of the aperture angles of a fusing wedge, is presented for exact solutions of the single-phase problem of a fusing wedge that were obtained earlier and are written for these values. The critical orientation of the fusing wedge to the axis of fusion when the written solution degenerates is indicated.

The present work is a continuation of [1, 2] and an addition to [2] and uses the main expressions and notation of these works.

1. In [2] it is shown that the solution to be composed contains a closing exponent containing the characteristic numbers α_n and β_n, provided that the point with the coordinates (α_n, β_n) is located on the characteristic ellipse:

$$Э(\alpha_n, \beta_n) = \alpha_n^2 + \beta_n^2 + 2B\alpha_n\beta_n - A_1\alpha_n - A_2\beta_n = 0, \tag{1}$$

where n is the number of terms in the spectra $\{\alpha_n\}$ and $\{\beta_n\}$.

The values of α_n and β_n can be expressed in the form of homogeneous dependences on A_1, A_2:

$$\alpha_n = A_1 P_n - A_2 Q_n, \quad \beta_n = -A_1 Q_n + A_2 P_n, \tag{2}$$

where P_n and Q_n are polynomials that depend on B.

Using expressions (2) and the auxiliary polynomials T_n and S_n, the authors were able to express the closing condition (1) in [2] in the form of polynomial equations for determination of B:

$$P_n = Q_n, \quad (P_n - Q_n)(1 + B) = 1. \tag{3}$$

Let us consider a particular example of realization of conditions of closing (3). Let $n = 6$; the angles corresponding to five roots of the first condition in (3) and to six roots of the second condition in it, will be the following, respectively:

$$\frac{2\pi}{6}, \frac{4\pi}{6}, \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \quad \text{and} \quad \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}. \tag{4}$$

If the closing conditions (3) are written for the values from $n = 1$ to $n = 6$ in succession, then it can be seen that two groups of angles satisfy conditions (3) for a given value of n. One group of angles corresponds to the value of n taken, and the other recurs, having appeared first in the condition written for $n-1$. The angles in (4) for which the closing condition for $n = 6$ is satisfied contain a group of angles satisfying the closing conditions for $n = 5$:

$$\frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}. \tag{5}$$

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This ambiguity of the closing conditions (3) can be eliminated. If the point (α_n, β_n), whose coordinates have the number n, is a closing point on the characteristic ellipse, then its coordinates and the coordinates of the two adjacent points (α_n, β_{n-1}) and (α_{n-1}, β_n) are related by the conditions

$$Э(\alpha_n, \beta_{n-1}) = 0, \quad Э(\alpha_{n-1}, \beta_n) = 0. \tag{6}$$

From conditions (6) and (1) with allowance for (2) we obtain conditions in the form of the polynomial equations:

$$P_n + P_{n-1} - 2BQ_n = 1, \quad Q_n + Q_{n-1} - 2BP_n = 0. \tag{7}$$

As is mentioned in [2], the polynomials P_n and Q_n possess the property

$$P_{2m-1} = P_{2m}, \quad Q_{2m} = Q_{2m+1}, \quad m = 1, 2, \ldots \tag{8}$$

With account for the properties of polynomials (8), conditions (7) are divided into conditions for $n = 2m$ and $n = 2m + 1$:

$$2(P_{2m} - BQ_{2m}) = 1; \quad Q_{2m+1} - BP_{2m+1} = 0. \tag{9}$$

For even n, we write the first condition from (9), for odd n, we write the second condition. Conditions (9) are polynomial equations of degree n relative to B.

For B satisfying one of the conditions in (9), the point (α_n, β_n) is located on the characteristic ellipse and is a closing point in the computations. Conditions (9) will be called closing conditions.

The second condition from (9) will be composed for $n = 5$. The angles corresponding to five roots of B in this condition are written in (5). The first condition from (9) will be composed for $n = 6$. The angles corresponding to six roots of B in this condition will take the form

$$\frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}.$$

The closing conditions (9) give an unambiguous result. According to the value of n taken, angles of the form $\psi = \pi m/n + 1$, $m = 1, 2, \ldots, n$ satisfy the closing conditions in (9).

2. Let us consider the case where the angle $\psi = \pi m/n + 1$ between the normals to the fusing surfaces, which satisfies the closing condition (9) for the given value of n, is composed of smaller fractions of division π, $\psi_k = \pi mk/(n + 1)k$, $k = 1, 2, \ldots$. The angle ψ_k satisfies different closing conditions for the various values of the factor k.

Calculations by recurrence formulas (11) from [2] are performed from the point $(\alpha_{i-2}, \beta_{i-1})$ to the point (α_i, β_{i-1}), and further to the point (α_i, β_{i+1}), ... Having connected these points by line segments, we call the resultant polygonal line a characteristic polygonal line. Two characteristic polygonal lines originate from the point (0, 0) and contact at the closing point (α_n, β_n).

Since the value of $B = \cos \psi_k$ determining the characteristic ellipse (9) from [2] is unique for all values of k, calculations by recurrence formulas (11) from [2] for $k > 1$ are continued beyond the number n, repeating the values of the spectra $\{\alpha_n\}$, $\{\beta_n\}$ obtained before the number n. At the closing point (α_n, β_n) the characteristic polygonal lines for $k = 2$ coincide and return to the point (0, 0). For $k = 3$, at the point (0, 0) the characteristic polygonal lines coincide again and contact at the closing point (α_n, β_n) once more. Thus, the values of the spectra $\{\alpha_n\}$, $\{\beta_n\}$ determined for $k = 1$ will be repeated k times in the spectra calculated for $k > 1$, and sum (26) in [2] for spectra repeating their values k times is composed from the k-times repeated sum of the exponents that represents the solution for the angle ψ_k at $k = 1$. But the sum of k solutions is not a solution since at infinitely remote points of the fusing wedge, the sum of k solutions tends to kU_∞ rather than to U_∞, i.e., condition (3) from [2] is not satisfied. Having verified that this condition is satisfied, it is possible to determine the value of the random factor k.

Thus, the angles ψ_k for $k > 1$ satisfy correctly the various closing conditions (9) for the various values of the factor k, but sum (26) from [2], formally composed for the angle ψ_k, is not a solution. An exact solution of (26) form from [2], unique for each value of the angle, is written at $k = 1$, i.e., for the case when the fraction $m/(n + 1)$, determining the value of the angle ψ, is simple.

3. The fusing wedge can be oriented to the axis of fusion so that a solution of the form of (26) from [2] degenerates. This is the case when the characteristic polygonal line occurs at the extremum point of characteristic ellipse (9) from [2].

The extremum condition is written, for example, for α as

$$A_2 - 2\beta - 2B\alpha = 0\,. \tag{10}$$

For the characteristic numbers α_k, β_{k-1}, composing the coordinates of the point (α_k, β_{k-1}) that satisfies condition (10), the following condition is valid:

$$\beta_{k-1} = A_2 - \beta_{k-1} - 2B\alpha_k\,. \tag{11}$$

Then from recurrence relations (11) from [2] with account for (11), we calculate $\beta_{k+1} = \beta_{k-1}$, $\alpha_{k+2} = \alpha_{k-2}$, ... This means that after reaching the extremum point on the ellipse, the characteristic numbers α_k and β_k begin to repeat their previous values in the inverse order. From the extremum point the characteristic polygonal line returns toward the point (0, 0). One of the intermediate points of this polygonal line is the closing point (α_n, β_n). The second characteristic polygonal line originating from the point (0, 0) occurs at the closing point (α_n, β_n) only having passed the same route, i.e., having returned from its extremum point. The characteristic polygonal lines pass twice through each point, and sum (26) from [2] appears to be composed of pairs of the same exponents with different signs and becomes zero, i.e., the solution degenerates.

The form of the characteristic ellipse (9) from [2] is determined by the value of $B = \cos\psi$, and the angle between its axes and the axes of the coordinates α, β is always $\pi/4$. Then, for the prescribed value of B the positions of the extremum points on the ellipse are fixed and the characteristic polygonal lines passing through them are unique.

At the prescribed B, changes in A_1 and A_2 for the ellipse equation mean parallel transfer of the axes of the coordinates α, β when their origin slides along the ellipse line.

To determine the lengths of the line segments δ_n composing the characteristic polygonal lines that connect the two extremum points, the origin of the coordinates α and β will be placed at the point on the ellipse where $A_2 = 0$. This point is the extremum point for α, because its coordinates (0, 0) satisfy condition (10). Then we determine the lengths of the line segments δ_n composing the characteristic polygonal line that originates from this extremum point:

$$\delta_1 = |\alpha_1|\,, \quad \delta_{2m} = |\beta_{2m} - \beta_{2m-2}|\,, \quad \delta_{2m+1} = |\alpha_{2m+1} - \alpha_{2m-1}|\,, \quad m = 1, 2, \ldots$$

Bearing in mind that A_1, A_2, and B are related by a trigonometric formula and taking into account representation (2), the unknown lengths will be expressed in polynomial form:

$$\delta_1 = \sqrt{1 - B^2}\,, \quad \delta_{2m} = \delta_1\,|Q_{2m-2} - Q_{2m}|\,, \quad \delta_{2m+1} = \delta_1\,|P_{2m+1} - P_{2m-1}|\,. \tag{12}$$

Since the ellipse is symmetric, the values of δ_n have the property $\delta_n = \delta_1$, $\delta_{\nu-1} = \delta_2$, ... For a prescribed value of B the number of lengths δ_n coincides with the number of terms in the spectra $\{\alpha_n\}$, $\{\beta_n\}$.

With prescribed B changes in A_1 and A_2 mean a change in the orientation of the fusing wedge toward the axis of fusion z. When $A_1 = 0$ or $A_2 = 0$, the axis of fusion z is located along one of the faces of the fusing wedge. The pair of numbers (A_1, A_2) will be referred to as a direction of the axis of fusion. If after the direction of the axis of fusion changes, it turns out that $A_1 = \delta_k$ and, consequently, $A_2 = \delta_{k\pm 1}$, then the characteristic polygonal lines of ellipse (9) from [2] occur at the extremum points and the solution of the form of (26) from [2] degenerates. The direction of the axis of fusion at which the solution degenerates will be called critical.

For a fusing wedge that has the aperture angle $\pi m/n + 1$ between the normals to its faces, the pairs of the values $(0, \delta_1)$, (δ_1, δ_2), ..., (δ_{n-1}, δ_n), $(\delta_n, 0)$ determine $n + 1$ critical directions of the axis of fusion z.

The half-plane where the fusing wedge is located is divided by these directions into angular portions by the aperture $\pi/n + 1$. The wedge faces are determined by two of these directions: $(0, \delta_1)$ and $(\delta_n, 0)$. $m-1$ critical directions of the z axis occur inside the aperture of the wedge.

We consider an example. Let the aperture of the fusing wedge be $2\pi/3$; then $B = \cos(\pi/3) = 1/2$, $n = 2$. Using formulas (12), we calculate $\delta_1 = \delta_2 = \sqrt{3}/2$. Then, we obtain the critical directions of the z axis: $(0, \sqrt{3}/2)$, $(\sqrt{3}/2, \sqrt{3}/2)$, $(\sqrt{3}/2, 0)$, The first and last directions are directions of the faces of the wedge considered. The direction $(\sqrt{3}/2, \sqrt{3}/2)$ means a symmetric position of the z axis, dividing the aperture $2\pi/3$ into portions $\pi/3$. Indeed, if $A_1 = A_2 = \sqrt{3}/2$, then $\beta_n = \alpha_n$, and it is sufficient to determine just the spectrum $\{\alpha_n\} = \{\alpha_0 = 0, \alpha_1 = \sqrt{3}/2, \alpha_2 = 0\}$. Solution (26) from [2] composed for the spectrum obtained is degenerate.

Thus, the exact solution of the form of (26) from [2] exists only for cases where the axis of fusion z does not coincide with any of its critical directions. The critical directions of the axis of fusion are the lines dividing the fusing angle into portions. For the given B, the values of A_1 and A_2 should not coincide with any of the values of δ_n determined from (12).

NOTATION

α, β, α_n, β_n, δ_n, A_1, A_2, B, auxiliary variables; P_n, Q_n, S_n, T_n, polynomials; k, m, n, i, natural numbers; ψ, ψ_k, angles between the normals to the surfaces forming the fusing wedge; z, the axis in a Cartesian coordinate system; U_∞, temperature at points of the body infinitely remote from the fusing boundary.

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